Alice and Bob: a Love Story

Grenslops ♡

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December 9, 2019

Disclaimer

Any resemblance to figures real or imaginary is entirely coincidental.

First Sight

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She decides to play to her strengths, and use **MATH** to win Bob's affections.

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$B(t)$ will be Bob's position at the party, and $A(t) = (x(t), y(t))$ will be Alice's position. Alice will keep her eyes on the prize and move directly towards Bob at all times, not even stopping for an appetizer. She assumes Bob – consistent as he is – will move at a constant velocity v, and she will measure her speed as a constant multiple k of his.

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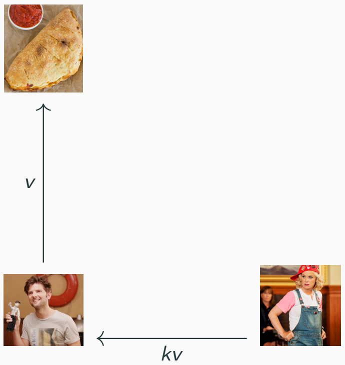

Knowing his love of calzones, Alice assumes Bob will make a beeline for the appetizers, which Alice models as Bob moving straight up the y-axis: his path is thus parametrized by $B(t) = (0, rt)$. Alice assumes she starts at some point $(c, 0)$.

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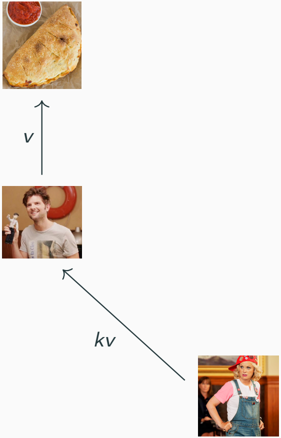

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With a bit of clever differentiation, chain rule, substitutions, and separable differential equation solving, Alice computes that her path is given by

$$y = \begin{cases} \frac{1}{2}\left[\frac{x^{1+1/k}}{c^{1/k}(1+1/k)} - \frac{c^{1/k}x^{1-1/k}}{1-1/k}\right] + \frac{ck}{k^2-1} & k \neq 1 \\ \frac{1}{2}\left[\frac{x^2-c^2}{2c} - c\ln\frac{x}{c}\right] & k = 1 \end{cases} .$$

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What if she only moves at his speed, so that $k = 1$? Then she calculates that

$$y_{Bob} - y_{Alice} = \frac{c}{2}\left[1 - \left(\frac{x}{c}\right)^2\right].$$

So in the limit ($x \to 0$), their difference is $\frac{c}{2}$. She only reaches halfway to him!

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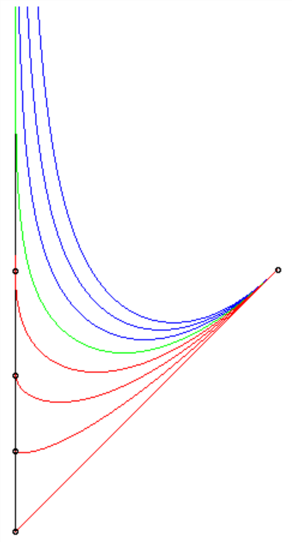

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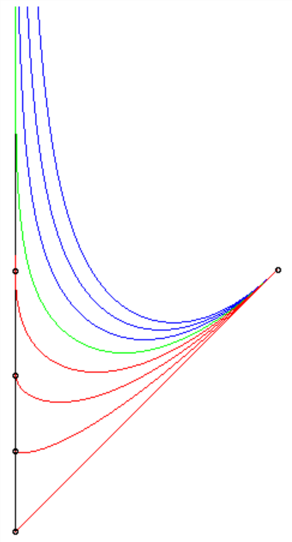

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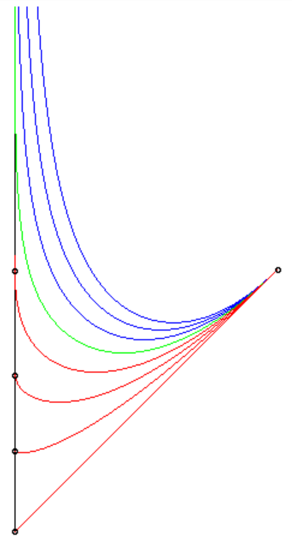

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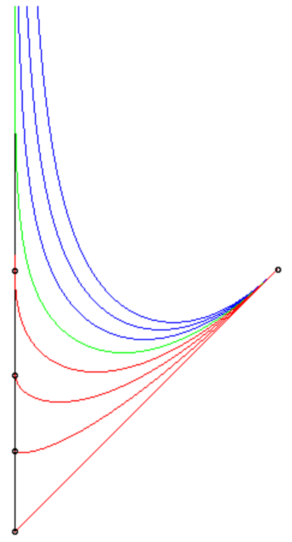

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Red means $k > 1$, and Alice catches up with Bob. Green means $k = 1$, and she never quite gets to him. Blue indicates Alice is moving too slowly, and Bob reaches the calzones before she can get to him.

A Setback

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The next day, she is dismayed to learn that Bob asked about her several times that night. He's never done this before, and Alice wonders if he's only interested when it seems like she's not. She decides to come up with a new model and pulls out her copy of Strogatz.

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for some $r, s > 0$, since Bob's affection waxes and wanes opposite of Alice's (she assumes he is a "Wave" in terms of attachment style).

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They will simultaneously be in love only a fourth of the time. Not good enough for Alice.

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(Hey Vance, write this on the board! ♡)

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- If $c_{B,B} < 0$ and $c_{B,A} > 0$, Bob is a cautious lover; will he and Alice ever make it work?

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Carol, who is not a mathematician and who knows how Alice often overthinks things, tells her to be direct and ask Bob on a date.

Courage

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Unfortunately, when they get their order there's a small problem: the sandwich maker has unevenly distributed the ham. Bob despairs, thinking one of them will get less of the sandwich than the other, but Alice remembers her topology class, and has just the theorem ready!

The Ham Sandwich Theorem

Theorem

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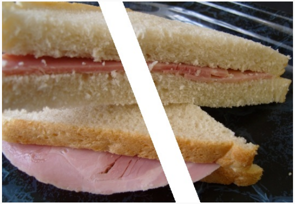

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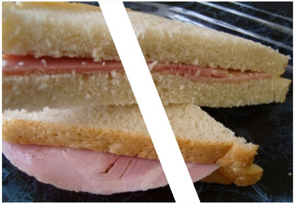

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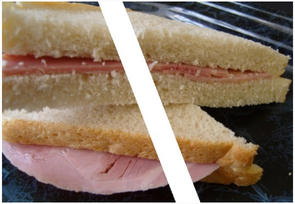

Alice explains to Bob that in three dimensions, they can think of the sandwich as three objects: bread + ham + bread, and evenly split it with a single cut. Bob is elated but skeptical and asks her to prove it. Here is Alice's proof.

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Define

$$\pi_i(p) = \text{volume of } A_i \text{ on the positive side of } H_p,$$

and let $f : S^{n-1} \to \mathbb{R}^{n-1}$ by

$$f(p) = (\pi_1(p), \pi_2(p), \dots, \pi_{n-1}(p)).$$

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Note that f is continuous!

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Recall the Borsuk-Ulam Theorem: if $f : S^{n-1} \to \mathbb{R}^{n-1}$ is continuous, then there exists $x \in S^{n-1}$ such that $f(-x) = f(x)$.

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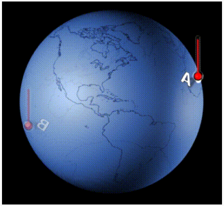

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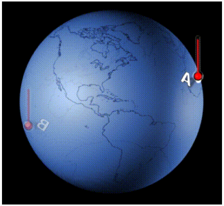

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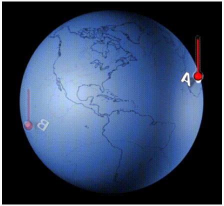

Applying Borsuk-Ulam to our setup, we have that there exists $p, q \in S^{n-1}$ such that $p = -q$ and $f(p) = f(q)$. But H_p and H_q are the same hyperplane, just with a different orientation, so $f(p) = f(q)$ implies that the volume of A_i is the same on both sides for all $i = 1, \ldots, n-1$. So H_p is the desired cut.

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Alice confidently cuts the sandwich in half, and both of them enjoy the rest of their first date.

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