Hyperelliptic Classes Are Rigid and Extremal in Genus Two

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Definitions

- The *cone of pseudo-effective classes of codimension* d on a projective variety X is denoted $\overline{\mathrm{Eff}}^d(X)$ (coefficients are taken in $\mathbb{R}$).
- An effective class $E \in \overline{\mathrm{Eff}}^d(X)$ is *rigid* if any effective cycle with class rE is supported on the support of E.
- An effective class $E \in \overline{\mathrm{Eff}}^d(X)$ is *extremal* if $E = E_1 + E_2$ implies E_1 is proportional to E.
- Denote by $\overline{\mathcal{H}}_{g,\ell,2m,n}$ the closure of the locus of hyperelliptic curves in $\overline{\mathcal{M}}_{g,\ell+2m+n}$ with marked Weierstrass points $w_1, \ldots, w_\ell$; pairs of marked points $+_1, -_1, \ldots, +_m, -_m$ with $+_j$ and $-_j$ conjugate under the hyperelliptic map; and free marked points $p_1, \ldots, p_n$ with no additional constraints.

Main Theorem

For $\ell, m, n \geq 0$, the class of $\overline{\mathcal{H}}_{2,\ell,2m,n}$, if nonempty, is rigid and extremal in the cone of effective classes of codimension $\ell + m$ in $\overline{\mathcal{M}}_{2,\ell+2m+n}$.

Properties of $\overline{\mathcal{H}}_{g,\ell,2m,n}$

- For $g \geq 2$, $\operatorname{codim} \overline{\mathcal{H}}_{g,\ell,2m,n} = g - 2 + \ell + m$.
- Hyperelliptic loci are irreducible.
- Hyperelliptic classes are effective and tautological.
- $\pi_{w_i *}\left[\overline{\mathcal{H}}_{g,\ell,2m,n}\right] = (2g + 2 - (\ell - 1))\left[\overline{\mathcal{H}}_{g,\ell-1,2m,n}\right]$
- $\pi_{+_j *}\left[\overline{\mathcal{H}}_{g,\ell,2m,n}\right] = \left[\overline{\mathcal{H}}_{g,\ell,2(m-1),n+1}\right] = \pi^*_{-_j}\left[\overline{\mathcal{H}}_{g,\ell,2(m-1),n}\right]$
- With $a = 2g + 2 - (\ell - 1)$, the following diagram commutes:

$$\begin{array}{ccc} \left[\overline{\mathcal{H}}_{g,\ell,2m,n}\right] & \xrightarrow{\pi_{+_j *}} & \left[\overline{\mathcal{H}}_{g,\ell,2(m-1),n+1}\right] \\ \downarrow \pi_{w_i *} & & \downarrow \pi_{w_i *} \\ a\left[\overline{\mathcal{H}}_{g,\ell-1,2m,n}\right] & \xrightarrow{\pi_{+_j *}} & a\left[\overline{\mathcal{H}}_{g,\ell-1,2(m-1),n+1}\right] \end{array}$$

Idea of Proof

We establish rigidity and extremality for the divisorial base cases $\left[\overline{\mathcal{H}}_{2,1,0,n}\right]$ and $\left[\overline{\mathcal{H}}_{2,0,2,n}\right]$ using the Key Lemma. We then induct on codimension using the fact that hyperelliptic classes push onto (multiples of) hyperelliptic classes of lower codimension. Extra care is required in the codimension-two case.

Base Case $\left[\overline{\mathcal{H}}_{2,0,2,n}\right]$

For both base cases, we utilize the following lemma.

Key Lemma [2]. *Let D be an irreducible effective divisor in a projective variety X, and suppose that $\mathcal{C}$ is a moving curve in D satisfying $[D] \cdot [\mathcal{C}] < 0$. Then $[D]$ is rigid and extremal.*

We define a moving curve $\mathcal{C}$ in $\left[\overline{\mathcal{H}}_{2,0,2,n}\right]$ by fixing a general genus-two curve C with n free marked points $p_1, \ldots, p_n$ and varying a conjugate pair $(+, -)$. Then, using the identity [4]

$$\left[\overline{\mathcal{H}}_{2,0,2,0}\right] = -\lambda + \psi_+ + \psi_- - 3\delta_{2,\varnothing} - \delta_{1,\varnothing},$$

we compute

$$\begin{aligned} \left[\overline{\mathcal{H}}_{2,0,2,n}\right] \cdot [\mathcal{C}] &= \left[\overline{\mathcal{H}}_{2,0,2,0}\right] \cdot \pi_{p_1 *} \cdots \pi_{p_n *}[\mathcal{C}] \\ &= -2. \end{aligned}$$

Base Case $\left[\overline{\mathcal{H}}_{2,1,0,n}\right]$

In order to define a moving curve which intersects negatively with $\left[\overline{\mathcal{H}}_{2,1,0,n}\right]$, we use the theory of admissible covers [1] and the following diagram (note that the image of s is precisely $\overline{\mathcal{H}}_{2,1,0,n} \subset \overline{\mathcal{M}}_{2,1+n}$).

$$\begin{array}{ccc} \overline{Adm}_{2 \xrightarrow{2} 0, t_1, \ldots, t_6, u_{1\pm}, \ldots, u_{n\pm}} & \xrightarrow{s} & \overline{\mathcal{M}}_{2,1+n} \\ \downarrow c & & \\ \overline{\mathcal{M}}_{0,\{t_1,\ldots,t_6,u_1,\ldots,u_n\}} & & \\ \downarrow \pi_{t_6} & & \\ \overline{\mathcal{M}}_{0,\{t_1,\ldots,t_5,u_1,\ldots,u_n\}} & & \end{array}$$

Taking a point $[b_n]$ in $\overline{\mathcal{M}}_{0,\{t_1,\ldots,t_5,u_1,\ldots,u_n\}}$ allows us to define $[\mathcal{B}_n] = s_* c^* \pi^*_{t_6}[b_n]$, a moving curve in $\overline{\mathcal{H}}_{2,1,0,n}$. We then show that this gives a negative intersection with $\left[\overline{\mathcal{H}}_{2,1,0,n}\right]$ via intersection theory on the admissible covers space.

Inducting on Codimension

The bulk of the effort is in establishing extremality. Suppose $\left[\overline{\mathcal{H}}\right] = \left[\overline{\mathcal{H}}_{2,\ell,2m,n}\right]$ is given an effective decomposition so that

$$\left[\overline{\mathcal{H}}\right] = \sum_s a_s[X_s] + \sum_t b_t[Y_t],$$

where the $[X_s]$ vanish under pushforward by all π_{w_i} and π_{+_j} and the $[Y_t]$ vanish under none of these.

We induct by pushing forward by all w_i and $+_j$ forgetful morphisms:

$$\begin{array}{ccc} \left[\overline{\mathcal{H}}\right] & = & \sum_s a_s[X_s] + \sum_t b_t[Y_t] \\ \downarrow \pi_* & & \downarrow \pi_* \\ \left[\overline{\mathcal{H}}\right]' & = & \sum_s a_s \pi_*[X_s] \end{array}$$

By hypothesis, $\pi_*[X_s]$ is proportional to $\left[\overline{\mathcal{H}}\right]'$, so

$$X_s \subseteq \bigcap \pi^{-1}\overline{\mathcal{H}}'$$

and hence $[X_s]$ is proportional to $\left[\overline{\mathcal{H}}\right]$. Subtract and rescale so that

$$\left[\overline{\mathcal{H}}\right] = \sum_t b_t[Y_t],$$

a contradiction, since $\pi_*\left[\overline{\mathcal{H}}\right] \neq 0$.

A Hiccup in Codimension Two

When $\ell = 2$ and $m = 0$, our inductive process is more subtle, as we can only conclude that the support of some $[X_s]$ which survives pushforward is in the intersection of preimages of hyperelliptic loci under π_{w_1} and π_{w_2}, which allows for an extra non-hyperelliptic possibility, the generic element of which is shown in Figure 1. We use a different intersection-theoretic calculation and a secondary induction argument to rule out such a possibility, showing that this class cannot be in the effective decomposition.

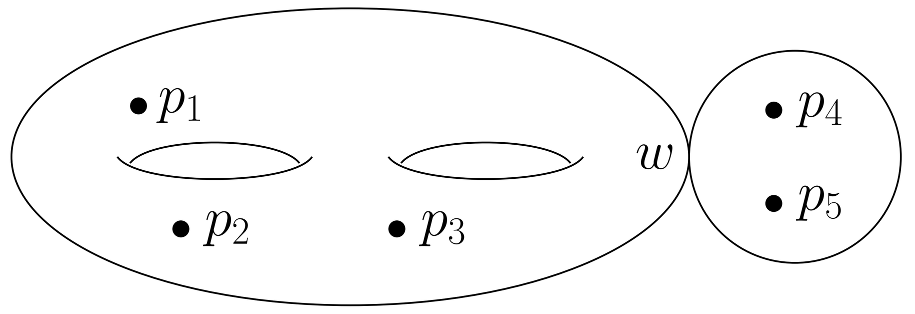

Figure 1: The hiccup for $\overline{\mathcal{H}}_{2,2,0,3}$.

Future Work

The induction process immediately generalizes for higher genus. However, the base cases in genus g are of codimension $g - 1$, and the techniques used for divisors do not extend to higher codimension. Moreover, the peculiarity of the $\ell = 2$ and $m = 0$ case will continue to be a problem in higher genus; to overcome this we rely on an explicit description of $\left[\overline{\mathcal{H}}_{2,2,0,0}\right]$, and there is no known explicit description of $\left[\overline{\mathcal{H}}_{g,2,0,0}\right]$ for arbitrary genus.

A potential avenue for tackling the higher genus base cases comes from a CohFT-like structure amongst the hyperelliptic classes. Though they do not form a proper CohFT, they appear to mimic one closely. Understanding this structure could lead to explicit descriptions or graph formulas for hyperelliptic classes.

References

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